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BRST Quantization of $N = 2$, 3, and 4 Superconformal Theories on Higher-Genus Riemann Surfaces

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Complex contour integral techniques, developed in a previous paper for the $N = 0$ and 1 superconformal theories on higher-genus Riemann surfaces, are applied to a Becchi–Rouet–Stora–Tyutin (BRST) quantization procedure of superconformal theories with $N = 2$, 3, and 4 super-Krichever–Novikov (KN) constraint algebras on a genus-g Riemann surface. The BRST charges of the superconformal theories are constructed and the nilpotency of the BRST charges is checked. The critical spacetime dimension and the "intercepts" are found for the $N = 2$ and 4 cases. Also calculated are the central charge and the "intercept" for the $N = 3$ case.

1. INTRODUCTION

BRST quantization is the fundamental approach to covariant quantization of general gauge theories and conformal theories. For early work on this subject, the reader can refer to Kuang (1992*a*) (henceforth referred to as paper I and whose equations will be quoted by their number preceded by I) and references therein. Recently, superconformal field theories on higher-genus Riemann surfaces (Eguchi and Ooguri, 1987; Bonora *et al.*, 1988*a,b*; Mezincescu *et al.*, 1989; Huang and Zhao, 1990; Dai *et al.*, 1990; Zucchini, 1991; Frau *et al.*, 1991; Xu, 1991) have attracted a lot of interest. The superconformal invariance in superconformal theories on higher-genus Riemann surfaces is a basic symmetry, called the super-Krichever–Novikov (KN) symmetry (Krichever and Novikov, 1987; Konisi *et al.*, 1989). The superalgebras corresponding to the super-KN symmetry are called as super-KN algebras. Super-KN algebras which are physically

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interesting are $N=0$, 2, 3, and 4 super-KN algebras. In paper I, we developed a complex contour integral method for the BRST quantization of superconformal theories on higher-genus Riemann surfaces in which a regularization prescription is implemented implicitly by the method of analytic continuation. The method essentially amounts to a clear way of handling the operator ordering. We have used the method to deal with the BRST quantization procedure of superconformal theories on higher-genus Riemann surfaces with the $N=0$ and 1 super-KN constraint algebras. It has been shown that the nilpotency of the BRST charge requires the critical spacetime dimension D to be 26 and "intercept" $\alpha(g)$ to be $1-\frac{3}{4}g-(9/8)g^2$ for the $N=0$ case, and 10 and $\frac{1}{2}-\frac{3}{4}g-(17/16)g^2$ for the $N=1$ case. The purpose of the present paper is to apply this method to the BRST quantization procedure of superconformal theories on higher-genus Riemann surfaces with the $N=2$, 3, and 4 super-KN constraint algebras. We will construct the BRST charges corresponding to the $N=2$, 3, and 4 cases and check the nilpotency of the BRST charges, which is a crucial test of the self-consistency of the quantization procedure as well as the quantum self-consistency of the super-KN algebras. The critical spacetime dimension or the central charge and the "intercepts" for these cases will be followed from the nilpotent condition of the BRST charges.

2. THE $N=2$ SUPER-KN CONSTRAINED SYSTEM

Consider a compact Riemann surface Σ of genus g and two distinguished points in a general position on Σ. The $N=2$ super-KN algebra on Σ can be constructed by using the linear transformation method (Kuang, 1992*a*; Konisi *et al.*, 1989) and the operator production expansion approach (Kuang, 1992*b*). It contains the KN generator (L_n), two super-KN generators (G_r^i, $i=1,2$), and the $U(1)$ current generator (H_n). The $N=2$ super-KN algebra is

$$\begin{aligned}
&[L_n, L_m] = (m-n)L_{n+m-g_0} + \tfrac{1}{4}D(n-g_0+1)(n-g_0)(n-g_0-1)\delta_{n+m,2g_0} \\
&[L_n, G_r^i] = (\tfrac{1}{2}n - r + \tfrac{1}{4}g)G^i_{n+r-g_0}, \qquad [L_n, H_m] = -(m-\tfrac{1}{2}g)H_{n+m-g} \\
&\{G_r^i, G_s^i\} = 2L_{r+s-\frac{1}{2}g} + D[(r-g)^2-\tfrac{1}{4}]\delta_{r+s,2g} \qquad \text{(no sum on } i) \qquad (1) \\
&\{G_r^1, G_s^2\} = 2i(r-s)H_{r+s-g}, \qquad [H_n, H_m] = \tfrac{1}{4}D(n-\tfrac{1}{2}g)\delta_{n+m,g} \\
&[H_n, G_r^1] = \tfrac{1}{2}iG^2_{n+r-\frac{1}{2}g}, \qquad [H_n, G_r^2] = -\tfrac{1}{2}iG^1_{n+r-\frac{1}{2}g}
\end{aligned}$$

where $g_0=\frac{3}{2}g$, g is the genus number of the Riemann surface, and D is the spacetime dimension. n, m, r, and s take integer (half-integer) values when g is even (odd). The BRST charge corresponding to the algebra (1) can be written as

$$Q = Q_{00} + Q_{01} + Q_1 + \sum_{i=1}^{2} (Q_{02}^i + Q_2^i + Q_3^i + Q_6^i) + Q_{03} + Q_4 + Q_5 \quad (2)$$

with

$$Q_{00} = \sum_n L_n \eta_{-n} \quad (3)$$

$$Q_{01} = -\alpha(g)\eta_{-g_0} \quad (4a)$$

$$Q_1 = \tfrac{1}{2} \sum_{n,m} (n-m) :P_{n+m-g}\eta_{-m}\eta_{-n}: \quad (5a)$$

$$Q_2^i = \sum_{n,r} (\tfrac{1}{2}n - r + \tfrac{1}{4}g) :R_{n+r-g}^i \rho^i_{-r}\eta_{-n}: \quad (6a)$$

$$Q_{02}^i = \sum_r G_r^i \rho^i_{-r} \quad (7)$$

$$Q_3^i = -\sum_{r,s} :P_{r+s-\frac{1}{2}g}\rho^i_{-s}\rho^i_{-r}: \quad (8a)$$

$$Q_{03} = \sum_n H_n \hat{\eta}_{-n} \quad (9)$$

$$Q_4 = -2i \sum_{r,s} (r-s) :\hat{P}_{r+s-g}\rho^2_{-s}\rho^1_{-r}: \quad (10a)$$

$$Q_5 = -\sum_{n,m} (m - \tfrac{1}{2}g) :\hat{P}_{n+m-g_0}\hat{\eta}_{-m}\hat{\eta}_{-n}: \quad (11a)$$

$$Q_6^1 = \tfrac{1}{2}i \sum_{n,r} :R_{n+r-\frac{1}{2}g}^2 \rho^1_{-r}\hat{\eta}_{-n}: \quad (12a)$$

$$Q_6^2 = -\tfrac{1}{2}i \sum_{n,r} :R_{n+r-\frac{1}{2}g}^1 \rho^2_{-r}\hat{\eta}_{-n}: \quad (13a)$$

where η_{-n} and P_n are, respectively, the ghost and antighost corresponding to the constraints L_n, whereas ρ^i_{-r} and R_r^i, and $\hat{\eta}_{-n}$ and $\hat{P}_n$, are, respectively, the ghost and antighost corresponding to the constraints G_r^i and H_n. They satisfy the relations $\{\eta_n, P_m\} = \delta_{n+m,0}$, $\{\hat{\eta}_n, \hat{P}_m\} = \delta_{n+m,0}$, and $[\rho_r^i, R_s^j] = \delta^{ij}\delta_{r+s,0}$. The η_{-g_0} term in (4) comes out through normal ordering of the "energy operator" L_{g_o}; following paper I, the parameter $\alpha(g)$, which is undetermined so far, is called the "intercept."

We then express terms on the rhs of equation (2) as complex contour integrals:

$$Q_{01} = -\alpha(g) \oint dz\, z^{-g_0-1}\eta(z) \tag{4b}$$

$$Q_1 = -\oint dz\, z^{-g_0} :P(z)\eta'(z)\eta(z): \tag{5b}$$

$$Q_2^i = \tfrac{1}{4}\oint dz\, z^{-g_0} :\left[2R^i(z)\rho^i(z)\eta'(z) - 4R^{i\prime}(z)\rho^i(z)\eta(z) + \frac{g}{z} R^i(z)\rho^i(z)\eta(z) \right]: \tag{6b}$$

$$Q_3^i = -\oint dz\, z^{-\frac{1}{2}g-1} :P(z)\rho^i(z)\rho^i(z): \tag{8b}$$

$$Q_4 = -2i \oint dz\, z^{-g_0} :\hat{P}(z)\rho^2(z)\rho^{1\prime}(z) - \hat{P}(z)\rho^{2\prime}(z)\rho^1(z): \tag{10b}$$

$$Q_5 = \oint dz\, z^{-g_0} :[\hat{P}(z)\hat{\eta}'(z)\eta(z) + \tfrac{1}{2}gz^{-1}\hat{P}(z)\hat{\eta}(z)\eta(z)]: \tag{11b}$$

$$Q_6^1 = \tfrac{1}{2}i \oint dz\, z^{-\frac{1}{2}g-1} :R^2(z)\rho^1(z)\hat{\eta}(z): \tag{12b}$$

$$Q_6^2 = -\tfrac{1}{2}i \oint dz\, z^{-g-1} :R^1(z)\rho^2(z)\hat{\eta}(z): \tag{13b}$$

where we have used the notations $\eta'(z) = d\eta(z)/dz$, $P(z) = \sum_n P_n z^{-n}$, and so on.

In order to check the nilpotency of the BRST charge, we have to evaluate the anticommutator $\{Q, Q\}$. From (2), we see that $\{Q, Q\}$ can be written as the sum of 105 anticommutators, so that we must evaluate the 105 anticommutators. By a careful analysis, it can be shown that $\{Q, Q\}$ can be expressed as

$$\{Q, Q\} = \{Q, Q\}_{\eta\eta} + \{Q, Q\}_{\rho\rho} + \{Q, Q\}_{\hat{\eta}\hat{\eta}} \tag{14}$$

$\{Q, Q\}_{\eta\eta}$ in (14) is the contribution of all anticommutators to the KN ghost bilinear terms,

$$\{Q, Q\}_{\eta\eta} = \{Q_{00}, Q_{00}\} + (Q_1, Q_1\} + \sum_{i=1}^{2} \{Q_2^i, Q_2^i\} + \{Q_5, Q_5\} + 2\{Q_{01}, Q_1\} \tag{15}$$

The contribution to the terms of the form $\rho^i_{-r}\rho^i_{r-2g}$ comes from the anticommutators

$$\{Q, Q\}_{\rho\rho} = \sum_i \{Q^i_{02}, Q^i_{02}\} + 2\sum_{i,j} \{Q^i_2, Q^j_3\} + 2\sum_i \{Q_4, Q^i_6\} + 2\{Q_3, Q_{01}\} \tag{16}$$

The contribution to the ghost bilinear terms $\hat{\eta}_{-n}\hat{\eta}_{n-g}$ comes from

$$\{Q, Q\}_{\hat{\eta}\hat{\eta}} = \{Q_{03}, Q_{03}\} + 2\{Q^1_6, Q^2_6\} \tag{17}$$

We now evaluate the anticommutators on the rhs of equations (15)–(17).

2.1. The KN Ghost Bilinear Terms

For convenience, we first calculate $\{Q_5, Q_5\}$ in (15). According to the analysis in paper I, the anticommutator for two Q_5 can be expressed as the following complex double contour integral:

$$\{Q_5, Q_5\} = \tfrac{1}{2}\oint_{C_0}\oint_{C_w} dz\, dw\, (zw)^{-g} :[2\hat{P}(z)\hat{\eta}'(z)\eta(z) - gz^{-1}\hat{P}(z)\hat{\eta}(z)\eta(z)]:$$
$$\times :[\hat{P}(w)\hat{\eta}'(w)\eta(w) - gw^{-1}\hat{P}(w)\hat{\eta}(w)\eta(w)]: \tag{18a}$$

where C_w is a contour (of z) around $z = w$ and C_0 (of w) around the origin (Fig. 1).

To evaluate (18a), we do the Wick contraction of the operator products in the integrands. In this way we isolate the singularity of the integrands in the limit $z \to w$.

After using the Wick contraction theorem for the integrands in (18a), one obtains four terms with zero contraction, eight terms with one contraction, and four terms with two contractions. Because all those terms with zero contraction do not have any singularity at $z = w$, they have no

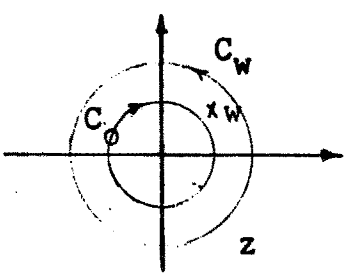

Fig. 1. The choice of contour for the complex integral in equation (18a).

contribution to the integral (18a). Substituting the ghost propagators

$$P(z)\eta(w) = \frac{z}{z-w} = \hat{P}(z)\hat{\eta}(w), \qquad \eta(z)P(w) = -\frac{w}{w-z} = \hat{\eta}(z)\hat{P}(w), \qquad |z| > |w| \tag{19}$$

into the terms with one concentration, after doing the z integration and using the properties that $\eta^2 = 0 = (\eta')^2$, one can see that the integrals of all one-contraction terms in (18a) vanish. The only terms that give nonzero contributions are those with two contractions; therefore equation (18a) can be written as

$$\begin{aligned}
\{Q_5, Q_5\} = \frac{1}{4}\oint_{C_0}\oint_{C_w} dz\, dw\, (zw)^{-g}[4:\hat{P}(z)\hat{\eta}'(z)\eta(z)\hat{P}(w)\hat{\eta}'(w)\eta(w):& \\
- 2gw^{-1}:\hat{P}(z)\hat{\eta}'(z)\eta(z)\hat{P}(w)\hat{\eta}(w)\eta(w):& \\
- 2gz^{-1}:\hat{P}(z)\hat{\eta}(z)\eta(z)\hat{P}(w)\hat{\eta}'(w)\eta(w):& \\
+ g^2(zw)^{-1}:\hat{P}(z)\hat{\eta}(z)\eta(z)\hat{P}(w)\hat{\eta}(w)\eta(w):]& \\
= I_1 + I_2 + I_3 + I_4&
\end{aligned} \tag{18b}$$

Substituting the mode expansions of ghost fields and (19) into (18b), we have

$$\begin{aligned}
I_1 &= -\sum_{n,m}\oint_{C_0}\oint_{C_w} dz\, dw\, \frac{z^{n-g_0}w^{m-g_0+1}}{(z-w)^4}\eta_{-n}\eta_{-m} \\
&= -\frac{1}{6}\sum_n (n-g_0+1)(n-g_0)(n-g_0-1)\eta_{-n}\eta_{n-2g_0} \\
I_2 &= \tfrac{1}{2}g\sum_{n,m}\oint_{C_0}\oint_{C_w} dz\, dw\, \frac{z^{n-g_0+1}w^{m-g_0}}{(z-w)^3}\eta_{-n}\eta_{-m} \\
&= \tfrac{1}{4}g\sum_n (n-g_0+1)(n-g_0)\eta_{-n}\eta_{n-2g_0} \\
I_3 &= -\tfrac{1}{2}g\sum_{n,m}\oint_{C_0}\oint_{C_w} dz\, dw\, \frac{z^{n-g_0}w^{m-g_0+1}}{(z-w)^3}\eta_{-n}\eta_{-m} \\
&= -\tfrac{1}{4}g\sum_n (n-g_0)(n-g_0-1)\eta_{-n}\eta_{n-2g_0} \\
I_4 &= \tfrac{1}{4}g^2\sum_{n,m}\oint_{C_0}\oint_{C_w} dz\, dw\, \frac{z^{n-g_0}w^{m-g_0}}{(z-w)^2}\eta_{-n}\eta_{-m} \\
&= -\tfrac{1}{4}g^2\sum_n (n-g_0)\eta_{-n}\eta_{n-2g_0}
\end{aligned}$$

Therefore,

$$\{Q_5, Q_5\} = -\frac{1}{18}\sum_n [3n^3 - 9g_0 n^2 - (7g_0^2 - 6g_0 - 3)n - g_0(g_0^2 - 6g_0 - 3)]\eta_{-n}\eta_{n-2g_0} \tag{18c}$$

Then we evaluate the anticommutators of two Q_2^i,

$$\begin{aligned}\{Q_2^i, Q_2^i\} = \frac{1}{16}\oint_{C_0}\oint_{C_w} dz\, dw\, (zw)^{-g_0} &:[2R^i(z)\rho^i(z)\eta'(z) - 4R^i(z)\rho^{i\prime\prime}(z)\eta(z) \\ &- gz^{-1}R^i(z)\rho^i(z)\eta(z)]: :[2R^i(w)\rho^i(w)\eta'(w) - 4R^i(w)\rho^{i\prime\prime}(w)\eta(w) \\ &- gw^{-1}R^i(w)\rho^i(w)\eta(w)]:\end{aligned} \tag{20a}$$

Using Wick's theorem for the above integrand, one obtains nine terms without any contraction, 18 terms with one contraction, and 9 terms with two contractions. Again we can show that terms with less than two contractions do not contribute. The terms with two contractions can be written as

$$\begin{aligned}\{Q_2^i, Q_2^i\} = \frac{1}{16}\oint_{C_0}\oint_{C_w} dz\, dw\, (zw)^{-g_0}[&4:R^i(z)\rho^i(z)\eta'(z)R^i(w)\rho^i(w)\eta(w): \\ &- 8:R^i(z)\rho^i(z)\eta'(z)R^i(w)\rho^{i\prime\prime}(w)\eta(w): \\ &- 2g:R^i(z)\rho^i(z)\eta'(z)R^i(w)\rho^{i\prime\prime}(w)\eta(w): \\ &- 8:R^i(z)\rho^{i\prime\prime}(z)\eta(z)R^i(w)\rho^i(w)\eta'(w): \\ &+ 16:R^i(z)\rho^{i\prime\prime}(z)\eta(z)R^i(w)\rho^{i\prime\prime}(w)\eta(w): \\ &+ 4gw^{-1}:R^i(z)\rho^{i\prime\prime}(z)\eta(z)R^i(w)\rho^i(w)\eta(w): \\ &- 2gz^{-1}:R^i(z)\rho^i(z)\eta(z)R^i(w)\rho^i(w)\eta'(w): \\ &+ 4gz^{-1}:R^i(z)\rho^{i\prime\prime}(z)\eta(z)R^i(w)\rho^{i\prime\prime}(w)\eta(w): \\ &+ g^2(zw)^{-1}:R^i(z)\rho^i(z)\eta^i(z)R^i(w)\rho^i(w)\eta(w): \\ = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 &\end{aligned} \tag{20b}$$

Using the ghost propagators

$$\rho^i(z)R^i(w) = -R^i(z)\rho^i(w) = \frac{zw}{z-w}, \qquad |z|>|w| \tag{21}$$

after a lengthy calculation, we obtain

$$\begin{aligned}
I_1 &= \frac{1}{4}\sum_n n(n-g_0)(n-2g_0)\eta_{-n}\eta_{n-2g_0} \\
I_2 &= \frac{1}{4}\sum_n n(n-g_0)^2\eta_{-n}\eta_{n-2g_0} \\
I_3 &= \frac{1}{8}\sum_n n(n-g_0)\eta_{-n}\eta_{n-g_0} \\
I_4 &= -\frac{1}{4}\sum_n (n-2g_0)(n-g_0)^2\eta_{-n}\eta_{n-2g_0} \\
I_5 &= \frac{1}{12}\sum_n [(n-g_0)+3(n-g_0)^3]\eta_{-n}\eta_{n-2g_0} \\
I_6 &= -I_8 = \tfrac{1}{12}g_0\sum_n (n-g_0)^2\eta_{-n}\eta_{n-2g_0} \\
I_7 &= -\tfrac{1}{8}g_0\sum_n (n-g_0)(n-2g_0)\eta_{-n}\eta_{n-2g_0} \\
I_9 &= \tfrac{1}{36}g_0^2\sum_n (n-g_0)\eta_{-n}\eta_{n-2g_0}
\end{aligned} \tag{22}$$

Therefore,

$$\{Q_2^i, Q_2^i\} = \frac{1}{36}\sum_n [3n^3 - 99g_0n^2 + (97g_0^2+3)n - (31g_0^3+3g_0)]\eta_{-n}\eta_{n-2g_0} \tag{20c}$$

The anticommutator of two Q has been evaluated in paper I. From (I.29), we have

$$\begin{aligned}
\{Q_1, Q_1\} = \frac{1}{6}\sum_n [&-13n^3 + 39g_0n^2 - (33g_0^2 - 6g_0 - 1)n \\
&+ g_0(7g_0^2 - 6g_0 - 1)]\eta_{-n}\eta_{n-2g_0}
\end{aligned} \tag{23}$$

The evaluation of other anticommutators in equation (15) is trivial; the results are

$$\{Q_{00}, Q_{00}\} = \tfrac{1}{4}D\sum_n (n+g_0+1)(n-g_0)(n-g_0-1)\eta_{-n}\eta_{n-2g_0} \tag{24}$$

$$\{Q_1, Q_{01}\} = \alpha(g)\sum_n (n-g_0)\eta_{-n}\eta_{n-2g_0} \tag{25}$$

Therefore, we add all the terms of the form $\eta_{-n}\eta_{-n}$ in $\{Q, Q\}$ to obtain

$$\{Q, Q\}_{\eta\eta} = \frac{1}{12}\sum_n \{(6-2D)n^3 + (18-2D)n^2 + [(9D-6)g_0^2 + g_0 + (6-3D) + 24\alpha(g)]n - [(6+2D)g_0^2 + 16g_0 + 24\alpha(g)]g_0\}\eta_{-n}\eta_{n-2g_0} \tag{26}$$

2.2. The Super-KN Ghost Bilinear Terms

The contribution to the terms of the form $\rho^i_{-r}\rho^i_{r-2g}$ $(i = 1, 2)$ comes from the anticommutators on the rhs of equation (16). The first two of these anticommutators have been evaluated in paper I. After contacting the ghost and antighost fields, we get for the third anticommutator (to be specific we take $i = 1$),

$$\{Q_4, Q_6^1\} = \oint_{C_0}\oint_{C_w} dz\, dw\, z^{-g_0} w^{-\frac{1}{2}g-1} :[\hat{P}(z)\rho^2(z)\rho^{1\prime}(z) - \hat{P}(z)\rho^{2\prime}(z)\rho^1(z)]: \times :R^2(w)\rho^1(w)\hat{\eta}(w): \tag{27a}$$

Again, all terms with less than two contractions can be shown to be zero. Then we have

$$\begin{aligned}\{Q_4, Q_6^1\} &= \oint_{C_0}\oint_{C_w} dz\, dw\, z^{-g_0} w^{-\frac{1}{2}g-1}[:\hat{P}(z)\rho^2(z)\rho^{1\prime}(z)R^2(w)\rho^1(w)\hat{\eta}(w): \\ &\quad - :\hat{P}(z)\rho^{2\prime}(z)\rho^1(z)R^2(w)\rho^1(w)\hat{\eta}(w):] \\ &= \frac{1}{2}\oint_{C_0}\oint_{C_w} dz\, dw\, \frac{z^{-g_0+\frac{1}{2}} w^{-\frac{1}{2}g-\frac{1}{2}}}{(z-w)^3} \\ &\quad \times [2z(z-w):\rho^{1\prime}(z)\rho^1(w): + (z+w):\rho^1(z)\rho^1(w):] \\ &= \frac{1}{72}\sum_r (108r^2 + 336g_0 r - 14g_0^2 + 18g_0 + 9)\rho^1_{-r}\rho^1_{r-2g}\end{aligned} \tag{27b}$$

From (I.15), we obtain

$$\{Q_2^1, Q_3^1\} = \frac{1}{24}\sum_r [-60r^3 - 80g_0 r + (6g_0^2 + 12g_0 + 3)]\rho^1_{-r}\rho^1_{r-2g} \tag{28}$$

One can get easily anticommutators in equation (16),

$$\{Q_{02}^1, Q_{02}^1\} = D\sum_r [(r - \tfrac{2}{3}g_0)^2 - \tfrac{1}{4}]\rho^1_{-r}\rho^1_{r-2g} \tag{29}$$

$$\{Q_3^1, Q_{01}^1\} = \alpha(g)\sum_r \rho^1_{-r}\rho^1_{r-2g} \tag{30}$$

Thus all the terms of the form $\rho^1_{-r}\rho^1_{r-2g}$ can be written as

$$\{Q, Q\}_{\rho^1\rho^1} = \frac{1}{36}\sum_r [(36D - 72)r^2 + (96 - 48D)g_0 r + (16D + 4)g_0^2 + 48g_0 + (18 - 9D) + 72\alpha(g)]\rho^1_{-r}\rho^1_{r-2g} \tag{31}$$

The coefficients for terms of the type $\rho^2_{-r}\rho^2_{r-2g}$ can be written exactly in the same way except for replacing Q^i_1 by Q^i_2.

2.3. The Terms of the Type $\hat{\eta}_{-n}\hat{\eta}_{n-g}$

The contributions to the ghost bilinear $\hat{\eta}_{-n}\hat{\eta}_{n-g}$ come from the two anticommutators on the rhs of equation (17),

$$\{Q^1_6, Q^2_6\} = \frac{1}{4}\oint_{C_0}\oint_{C_w} dz\, dw\, (zw)^{-\frac{1}{2}g-1} \times :R^2(z)\rho^1(z)\hat{\eta}(z)::R^1(w)\rho^2(w)\hat{\eta}(w): \tag{32a}$$

After considering the Wick contractions, one gets the nonzero parts of (32a),

$$\begin{aligned}\{Q^1_6, Q^2_6\} &= \frac{1}{4}\oint_{C_0}\oint_{C_w} dz\, dw\, (zw)^{-\frac{1}{2}g-1}:R^2(z)\rho^1(z)\hat{\eta}(z)R^1(w)\rho^2(w)\hat{\eta}(w): \\ &= -\frac{1}{2}\sum_n (n - \tfrac{1}{2}g)\hat{\eta}_{-n}\hat{\eta}_{n-g}\end{aligned} \tag{32b}$$

It is easy to obtain

$$\{Q_{03}, Q_{03}\} = \tfrac{1}{4}D\sum_n (n - \tfrac{1}{2}g)\hat{\eta}_{-n}\hat{\eta}_{n-g} \tag{33}$$

Hence,

$$\{Q, Q\}_{\hat{\eta}\hat{\eta}} = \tfrac{1}{4}(D - 2)\sum_n (n - \tfrac{1}{2}g)\hat{\eta}_{-n}\hat{\eta}_{n-g} \tag{34}$$

Therefore the nilpotency of the BRST charge from equations (26), (31), and (34) demands

$$D = 2, \qquad \alpha(g) = -g - \tfrac{9}{8}g^2 \tag{35}$$

3. THE $N = 3$ SUPER-KN CONSTRAINED SYSTEM

We now consider the system with the $N = 3$ super-KN constraint algebra. We will determine the central charge by the nilpotency requirement of the BRST charge. Our result is that the nilpotency requires the central charge term to be zero (i.e., $C = 0$) and the "intercept" $\alpha(g) = -\frac{1}{2} - \frac{3}{2}g - \frac{3}{4}g^2$. This super-KN algebra consists of KN generators (L_n), super-KN generators (G^i_r, $i = 1, 2, 3$) and (T_r), and KN Kac–Moody generators (H^i_m). The $N = 3$ super-KN algebra can be written as

$$
\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m-g_0} + \frac{C}{12}(n-g_0+1)(n-g_0)(n-g_0-1)\delta_{n+m,2g}\\
&[L_n, G^i_r] = (\tfrac{1}{2}n - r)G^i_{n+r-g_0}, \qquad [L_n, T_r] = -(\tfrac{1}{2}n + r + \tfrac{1}{2}g_0)T_{n+-g_0}\\
&[H^i_n, G^i_r] = (n - \tfrac{1}{2}g)T_{n+r-g_0}, \qquad [H^i_n, G^j_r] = i\varepsilon^{ijk}G^k_{n+r-\frac{1}{2}g}\\
&\{G^i_r, G^i_s\} = 2L_{r+s-\frac{1}{2}g} + \tfrac{1}{3}C[(r-g)^2 - \tfrac{1}{4}]\delta_{r+s,2g} \qquad \text{(no sum on } i\text{)}\\
&\{G^i_r, G^j_s\} = i(r-s)\varepsilon^{ijk}H^k_{r+s-g_0} \qquad (i \neq j)\\
&\{T_r, T_s\} = \tfrac{1}{3}C\delta_{r+s,0}, \qquad [T_r, H^i_n] = 0\\
&\{T_r, G^i_s\} = iH^i_{r+s-\frac{1}{2}g}, \qquad [L_n, H^i_m] = -(m - \tfrac{1}{2}g)H^i_{n+m-g_0}\\
&[H^i_n, H^j_m] = i\varepsilon^{ijk}H^k_{n+m-\frac{1}{2}g}, \qquad [H^i_n, H^i_m] = \tfrac{1}{3}C(n - \tfrac{1}{2}g)\delta_{n+m,g}
\end{aligned}
\tag{36}
$$

where C is the central charge of the super-KN algebra and ε^{ijk} is the Levi-Civita symbol.

The BRST charge for this system can be written as

$$
\begin{aligned}
Q = {}& Q_{00} + Q_{01} + Q_1 + \sum_{i=1}^{3} (Q^i_2 + Q^i_{02} + Q^i_3 + Q^i_{03} + q^i_4 + Q^i_5 + Q^i_6)\\
&+ \sum_{i,j} (Q^{ij}_4 + Q^{ij}_6) + q_2 + q_{02} + Q_7
\end{aligned}
\tag{37}
$$

where Q_{00}, Q_{01}, Q_1, Q^i_2, Q^i_{02}, Q^i_3, Q^i_{03}, and Q^i_5 have been defined in equations (3)–(12), and

$$
\begin{aligned}
q_2 &= -\sum_{n,r} (\tfrac{1}{2}n + r + \tfrac{1}{2}g_0) :\hat{T}_{n+r-g_0} t_{-r} \eta_{-n}:\\
&= -\tfrac{1}{2}\oint dz :z^{-g_0}[\hat{T}(z)t(z)\eta'(z) + 2\hat{T}(z)t'(z)\eta(z) + g_0 z^{-1}\hat{T}(z)t(z)\eta(z)]:
\end{aligned}
\tag{38}
$$

$$
q_{02} = \sum_r T_r t_{-r} \tag{39}
$$

$$
\begin{aligned}
Q^{ij}_4 &= -i\sum_{r,s} (r-s)\varepsilon^{ijk} :P^k_{r+s-g_0} \rho^j_{-s} \rho^i_{-r}:\\
&= -i\varepsilon^{ijk} \oint dz :z^{-g_0}[\hat{P}^k(z)\rho^j(z)\rho^{i\prime}(z) - \hat{P}^k(z)\rho^{j\prime}(z)\rho^i(z)]
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
q^i_4 &= -i\sum_{r,s} \hat{P}^i_{r+s-\frac{1}{2}g} \rho^i_{-s} t_{-r}\\
&= -i\oint dz :^{-\frac{1}{2}g-1}\hat{P}^i(z)\rho^i(z)t(z):
\end{aligned}
\tag{41}
$$

$$q_6^i = \sum_{n,r} (n - \tfrac{1}{2}g) :\hat{T}_{n+r-g_0}\rho^i_{-r}\hat{\eta}^i_{-n}:$$

$$= \frac{1}{3}\oint dz :z^{-g_0-1}[z\hat{T}(z)\rho^i(z)\hat{\eta}'(z) - g_0\hat{T}(z)\rho^i(z)\hat{\eta}(z)]: \tag{42}$$

$$Q_6^{ij} = i\sum_{n,r} \varepsilon^{ijk} :R^k_{n+r-\frac{1}{2}g}\rho^j_{-r}\hat{\eta}^i_{-n}:$$

$$= i\varepsilon^{ijk}\oint dz :z^{-\frac{1}{2}g-1}R^k(z)\rho^j(z)\hat{\eta}^i(z): \tag{43}$$

$$Q_7 = i\sum_{n,m} \varepsilon^{ijk} :\hat{P}^k_{n+m-\frac{1}{2}g}\hat{\eta}^j_{-m}\hat{\eta}^i_{-n}:$$

$$= i\varepsilon^{ijk}\oint dz :z^{-\frac{1}{2}g-1}\hat{P}^k(z)\hat{\eta}^j(z)\hat{\eta}^i(z): \tag{44}$$

We now check the nilpotency of the BRST charge.

3.1. The KN Ghost Bilinear Terms

The ghost contribution to the bilinears of the form $\eta_n\eta_{n-2g_0}$ comes from

$$\{Q,Q\}_{\eta\eta} = \{Q_{00},Q_{00}\} + \{Q_1,Q_1\} + \sum_{i=1}^{3}\{Q_2^i,Q_2^i\} + \sum_{i=1}^{3}\{Q_5^i,Q_5^i\}$$
$$+ 2\{Q_1,Q_0\} + \{q_2,q_2\} \tag{45}$$

The contribution from each term is given as

$$\begin{aligned}
&\text{(I)} \quad \frac{C}{12}\sum_n [n^3 - 9g_0n^2 + (3g_0^2 - 1)n + g_0(1 - g_0^2)]\eta_{-n}\eta_{n-2g_0}\\
&\text{(II)} \quad \frac{1}{6}\sum_n [-13n^3 + 39g_0n^2 - (33g_0^2 - 6g_0 - 1)n\\
&\qquad + g_0(7g_0^2 - 6g_0 - 1)]\eta_{-n}\eta_{n-2g_0}\\
&\text{(III)} \quad \frac{1}{12}\sum_n [33n^3 - 99g_0n^2 + (97g_0^2 + 3)n - (31g_0^3 + 3g_0)]\eta_{-n}\eta_{n-2g_0}\\
&\text{(IV)} \quad \frac{1}{6}\sum_n [-3n^3 + 9g_0n^2 - (7g_0^2 - 6g_0 - 3)n\\
&\qquad + g_0(g_0^2 - 6g_0 - 3)]\eta_{-n}\eta_{n-2g_0}\\
&\text{(V)} \quad 2\alpha(g)\sum_n (n - g)\eta_{-n}\eta_{n-2g_0}\\
&\text{(VI)} \quad \frac{1}{12}\sum_n [-n^3 + 3g_0n^2 - (9g_0^2 - 1)n + g_0(g_0^2 - 1)]\eta_{-n}\eta_{n-2g_0}
\end{aligned} \tag{46}$$

All the anticommutators in equation (45) have been evaluated before except for the last one. Even this anticommutator can be written down just by looking at the evaluation of $\{Q^i_{\frac{1}{2}}, Q^i_{\frac{1}{2}}\}$. This anticommutator was given in equation (20b),

$$\{Q^i_{\frac{1}{2}}, Q^i_{\frac{1}{2}}\} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 \tag{20b}$$

After considering Wick concentrations, comparing the integrands of $\{q_2, q_2\}$ with those of $\{Q^i_{\frac{1}{2}}, Q^i_{\frac{1}{2}}\}$, we obtain

$$\begin{aligned}\{q_2, q_2\} &= I_1 - I_2 - 3I_3 - I_4 + I_5 + 3I_6 - 3I_7 + 3I_8 + 9I_9 \\ &= \frac{1}{12}\sum_n [-n^3 + 3g_0 n^2 - (9g_0^2 - 1)n + g_0(g_0^2 - 1)]\eta_{-n}\eta_{n-2g_0}\end{aligned} \tag{47}$$

Adding all the ghost contributions in equations (46), we get

$$\begin{aligned}\{Q, Q\}_{\eta\eta} = \frac{1}{12}\sum_n \{&Cn^3 - 9Cn^2 + [C(3g_0^2 - 1) + 8g_0^2 + 24g_0 + 24\alpha(g) + 12]n \\ &+ g_0[C(1 - g_0^2) - 8g_0^2 - 24g_0 - 24\alpha(g) - 12]\}\eta_{-n}\eta_{n-2g_0}\end{aligned} \tag{48}$$

3.2. The Super-KN Ghost Bilinear Terms (Triplet)

The contribution to the $\rho^i_{-r}\rho^i_{r-2g}$-type term comes from

$$\begin{aligned}\{Q, Q\}_{\rho\rho} = \sum_i \{Q^i_{02}, Q^i_{02}\} + 2\sum_i \{Q^i_{\frac{1}{2}}, Q^i_{\frac{3}{2}}\} + 2\sum_i \{q^i_4, q^i_6\} \\ + 4\sum_{i,j} \{Q^{ij}_4, Q^{ij}_6\} + 2\{Q^i_{03}, Q_{01}\}\end{aligned} \tag{49}$$

All these anticommutators have been evaluated before except for the third one. As an example, we evaluate

$$\begin{aligned}\{q^1_4, q^1_6\} = -\frac{i}{3}\oint_{C_0}\oint_{C_w} dz\, dw\, z^{-\frac{1}{2}g-1} w^{-g_0}[3:&\underline{\hat{P}(z)\rho^1(z)\underline{t(z)\hat{T}}(w)\rho^1(w)\hat{\eta}'}(w): \\ &- gw^{-1}:\underline{\hat{P}(z)\rho^1(z)\underline{t(z)\hat{T}}(w)\rho^1(w)\hat{\eta}}(w): \\ = \frac{1}{72}\sum_r (-36r^2 - 432g_0 r + 68g_0^2 + 12g_0 - 9)&\rho^1_{-r}\rho^1_{r-2g}\end{aligned} \tag{50}$$

Hence the contributions from different terms in equations (49) are, respectively,

$$\text{(I)}\quad \frac{C}{12}\sum_r [4(r-g)^2-1]\rho^1_{-r}\rho^1_{r-2g}$$

$$\text{(II)}\quad \frac{1}{12}\sum_r [-60r^2-80g_0r+(6g_0^2+12g_0+3)]\rho^1_{-r}\rho^1_{r-2g}$$

$$\text{(III)}\quad \frac{1}{36}\sum_r (-36r^2-432g_0r+68g_0^2+12g_0-9)\rho^1_{-r}\rho^1_{r-2g} \tag{51}$$

$$\text{(IV)}\quad \frac{1}{18}\sum_r (108r^2+338g_0r-14g_0^2+12g_0+9)\rho^1_{-r}\rho^1_{r-2g}$$

$$\text{(V)}\quad 2\alpha(g)\sum_r \rho^1_{-r}\rho^1_{r-2g}$$

They add up to

$$\{Q,Q\}_{\rho^1\rho^1}=\frac{1}{108}\sum_r [36Cr^2-48Cg_0r+D(16g_0^2-g)+72g_0^2+216g_0 + 216\alpha(g)+108]\rho^1_{-r}\rho^1_{r-2g} \tag{52}$$

3.3. The Super-KN Ghost Bilinear Terms (Singlet)

The contribution to the $t_{-r}t_{-r}$-type terms comes from only one anticommutator,

$$\{q_{02},q_{02}\}=\sum_r \{T_r,T_s\}t_{-r}t_{-s} = \tfrac{1}{3}C\sum_r t_{-r}t_{-r} \tag{53}$$

3.4. The KN Kac–Moody Ghost Bilinear Terms

The contribution to the ghost bilinear $\hat{\eta}^i_{-n}\hat{\eta}^i_{n-g}$ comes from

$$\{Q,Q\}_{\hat{\eta}\hat{\eta}}=\sum_i \{Q^i_{03},Q^i_{03}\}+\sum_{i,j}\{Q^{ij}_6,Q^{ij}_6\}+\{Q_7,Q_7\} \tag{54}$$

The corresponding contributions to the ghost bilinears are

$$\{Q,Q\}_{\hat{\eta}\hat{\eta}}=\frac{C}{3}\sum_n (n-\tfrac{1}{2}g)\hat{\eta}^i_{-n}\hat{\eta}^i_n-2\sum_n (n-\tfrac{1}{2}g)\hat{\eta}^i_{-n}\hat{\eta}^i_{n-g} + 2\sum_n (n-\tfrac{1}{2}g)\hat{\eta}^i_{-n}\hat{\eta}^i_{n-g} \tag{55}$$

Therefore the nilpotency of the BRST charge from equations (48), (52), (53), and (55) implies

$$C = 0, \qquad \alpha(g) = -\tfrac{1}{2} - \tfrac{3}{2}g - \tfrac{3}{4}g^2 \tag{56}$$

4. THE $N = 4$ SUPER-KN CONSTRAINED SYSTEM

The $N = 4$ super-KN algebra can be written as (the repeated Greek indices are summed over)

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m-g_0} + \tfrac{1}{2}D(n-g_0+1)(n-g_0)(n-g_0-1)\delta_{n+m,2g_0} \\
&[L_n, G_r^i] = (\tfrac{1}{2}n - r - \tfrac{1}{4}g)G_{n+r-g_0} \qquad (i = 1, 2, 3, 4) \\
&\{G_r^i, G_s^i\} = 2L_{r+s-\frac{1}{2}g} + 2D[(r-g) - \tfrac{1}{4}]\delta_{r+s,2g} \qquad (\text{no sum on } i) \\
&\{G_r^\alpha, G_s^\beta\} = 2i(r-s)\varepsilon^{\alpha\beta\gamma}H^\gamma_{r+s}, \qquad \{G_r^4, G_s^\alpha\} = 2i(r-s)H^\alpha_{r+s} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\alpha \neq \beta, \quad \alpha, \beta = 1, 2, 3) \\
&[L_n, H_m^\alpha] = -(n - \tfrac{1}{2}g)H^\alpha_{n+m-g_0}, \\
&[H_n^\alpha, G_r^\beta] = \tfrac{1}{2}i\varepsilon^{\alpha\beta\gamma}G^\gamma_{n+r-\frac{1}{2}g} - \tfrac{1}{2}i\delta^{\alpha+\beta}G^4_{n+r-g_0} \\
&[H_n^\alpha, G_r^4] = \tfrac{1}{2}iG^\alpha_{n+r-\frac{1}{2}g}, \qquad [H_n^\alpha, H_m^\beta] = i\varepsilon^{\alpha\beta\gamma}H^\gamma_{n+m-\frac{1}{2}g} + \tfrac{1}{2}D(n-\tfrac{1}{2}g)\delta^{\alpha\beta}\delta_{n+m,g}
\end{aligned} \tag{57}$$

where D is the central extension of the super-KN algebra.

The BRST charge for this system can be written as

$$\begin{aligned}
Q = Q_{00} + Q_{01} + Q_1 + \sum_{i=1}^{4} (Q_{02}^i + Q_2^i + Q_3^i) + \sum_{\alpha=1}^{3} (Q_{03}^\alpha + Q_4^{4\alpha} + Q_5^\alpha) \\
+ \sum_{\alpha\neq\beta} Q_4^{\alpha\beta} + \sum_{\alpha,i} Q_6^{\alpha i} + Q_7
\end{aligned} \tag{58}$$

where Q_{00}, Q_{01}, Q_1, Q_2^i, Q_{02}^i, Q_3^i, Q_{03}^i, Q_5, and Q_7 are the same as for the $N = 3$ case, and

$$Q_4^{\alpha\beta} = -2i\sum_{r,s}(r-s)\varepsilon^{\alpha\beta\gamma} :\hat{P}^\gamma_{r+s}\rho^\beta_{-s}\rho^\alpha_{-r}: \tag{59}$$

$$Q_4^{4\alpha} = -i\sum_{r,s}(r-s) :\hat{P}^\alpha_{r+s}\rho^\alpha_{-s}\rho^4_{-r}: \tag{60}$$

$$Q_6^{\alpha\beta} = -\tfrac{1}{2}i\sum_{r,n} :(\delta^{\alpha\beta}R^4_{n+r-g_0} + \varepsilon^{\alpha\beta\gamma}R^\gamma_{n+r-\frac{1}{2}g})\rho^\beta_{-r}\hat{\eta}^\alpha_{-n}: \tag{61}$$

$$Q_6^{\alpha 4} = \tfrac{1}{2}i\sum_{r,n} :R^\alpha_{n+r-\frac{1}{2}g}\rho^4_{-r}\hat{\eta}^\alpha_{-n}: \tag{62}$$

Now we write down the contributions to the ghost bilinears.

4.1. The KN Ghost Bilinear Terms

Nontrivial contributions to the terms of the type $\eta_{-n}\eta_{n-2g_0}$ come from

$$\{Q,Q\}_{\eta\eta}=\underset{\text{(i)}}{\{Q_{00},Q_{00}\}}+\underset{\text{(ii)}}{\{Q_1,Q_1\}}+\underset{\text{(iii)}}{\sum_{i=1}^{4}\{Q_2^i,Q_2^i\}}+\underset{\text{(iv)}}{\sum_{\alpha=1}^{3}\{Q_5^\alpha,Q_5^\alpha\}}+\underset{\text{(v)}}{2\{Q_1,Q_{01}\}} \tag{63a}$$

All these anticommutators have been evaluated before. Nonzero contributions coming from the different terms above can be written as

(i) $\frac{1}{2}D\sum_n [n^3-3g_0n^2+(3g_0^2-1)n-g_0(g_0^2-1)]\eta_{-n}\eta_{n-2g_0}$

(ii) $\frac{1}{6}\sum_n [-13n^3+39g_0n^2-(33g_0^2-6g_0-1)n + g_0(7g_0^2-6g_0-1)]\eta_{-n}\eta_{n-2g_0}$

(iii) $\frac{1}{9}\sum_n [33n^3-99g_0n^2+(97g_0^2+3)n-(31g_0^2+3g_0)]\eta_{-n}\eta_{n-2g_0}$

(iv) $\frac{1}{6}\sum_n [-3n^3+9g_0n^2-(7g_0^2-6g_0-3)n + g_0(g_0^2-6g_0-3)]\eta_{-n}\eta_{n-2g_0}$

(v) $2\alpha(g)\sum_n (n-g_0)\eta_{-n}\eta_{n-2g_0}$

Thus the total contribution is

$$\begin{aligned}\{Q,Q\}_{\eta\eta}=\frac{1}{18}\sum_n \{&(9D+18)n^3-(27D+54)n^2+[(27D-74)g_0^2+36g_0\\&+36\alpha(g)+(18-9D)]n-[(9D+38)g_0^2\\&+36g_0-36\alpha(g)-(9D-18)]g_0\}\eta_{-n}\eta_{n-2g_0}\end{aligned} \tag{63b}$$

4.2. The Super-KN Ghost Bilinear Terms

The contribution to the terms $\rho^i_{-r}\rho^i_{r-2g}$ comes from

$$\{Q,Q\}_{\rho\rho}=\sum_{i=1}^{4}\{Q_{02}^i,Q_{02}^i\}+2\sum_{i=1}^{4}\{Q_2^i,Q_3^i\}+2\sum_{i,j,k}\{Q_4^{ij},Q_6^k\}+2\{Q_3^i,Q_{01}\} \tag{64a}$$

From (I.51), (I.52) and (27), one obtains the total contribution,

$$\{Q, Q\}_{\rho\rho} = \frac{1}{36}\sum_r [(72D + 144)r^2 - 192g_0 r + (32D + 104)g_0^2 + 72g_0 + 72\alpha(g) - (18D - 9)]\rho^i_{-r}\rho^i_{r-2g} \tag{64b}$$

4.3. The Terms of the Type $\hat{\eta}^\alpha_{-n}\hat{\eta}^\alpha_{n-g}$

The contributions to the coefficient of the terms of the type $\hat{\eta}^\alpha_{-n}\hat{\eta}^\alpha_{n-g}$ come from the anticommutators

$$\{Q, Q\}_{\hat{\eta}\hat{\eta}} = \underset{\text{(i)}}{\{Q_{03}, Q_{03}\}} + \underset{\text{(ii)}}{\sum_{i,j} \{Q_6^i, Q_6^j\}} + \underset{\text{(iii)}}{\{Q_7, Q_7\}} \tag{65a}$$

From the previous evaluation, one can obtain the nonzero contributions coming from the different terms above,

(i) $\frac{1}{2}D\sum_n (n - \frac{1}{2}g)\hat{\eta}^\alpha_{-n}\hat{\eta}^\alpha_{n-g}$

(ii) $-\sum_n (n - \frac{1}{2}g)\hat{\eta}^\alpha_{-n}\hat{\eta}^\alpha_{n-g}$

(iii) $2\sum_n (n - \frac{1}{2}g)\hat{\eta}^\alpha_{-n}\hat{\eta}^\alpha_{n-g}$

Hence the total contribution is

$$\{Q, Q\}_{\hat{\eta}\hat{\eta}} = \frac{1}{2}(D + 2)\sum_n (n - \frac{1}{2}g)\hat{\eta}^\alpha_{-n}\hat{\eta}^\alpha_{n-g} \tag{65b}$$

From (63b), (64b), and (65b) we conclude that the nilpotency of the BRST charge requires

$$D = 2, \qquad \alpha(g) = -1 - \tfrac{3}{2}g - \tfrac{5}{2}g^2 \tag{66}$$

5. CONCLUDING REMARKS

We have studied the BRST quantization procedure of the superconformal theories on a genus-g Riemann surface with the $N = 2$, 3, and 4 super-KN algebras by using the complex contour integral techniques developed in paper I. We have checked the nilpotency of the BRST charges corresponding to these superconformal theories. For the $N = 2$ case, the critical spacetime dimension and the "intercept" have been found to be $D = 2$ and $\alpha(g) = -g - \frac{9}{8}g^2$. For the $N = 4$ system, we have obtained

$D = -2$ and $\alpha(g) = -1 - \frac{3}{2}g - \frac{5}{2}g^2$. These results can be related to the physical parameters (such as the number of scalar fields) in the Lagrangians, and they may be useful for the study of representation theories of the super-KN algebras (Kuang, 1992*b*). For the $N = 3$ case, we have found the central charge $C = 0$ and $\alpha(g) = -\frac{1}{2} - \frac{3}{2}g - \frac{3}{4}g^2$. Although the physical application of these results is not clear for the $N = 3$ case because of the lack of a Lagrangian realization, it may be useful at least in statistical mechanics or string field theories on higher-genus Riemann surfaces. It is obvious that when $g = 0$, our results can recover the known results (Qiu, 1987; Chang and Kumar, 1987; Kent and Riggs, 1987) on a trivial Riemann surface.

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